EXERCISES WEEK 1: AFFINE VARIETIES AND COORDINATE RINGS

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Exercise 1. Consider the ideal

$$J = (x^3z^2 - 2x^3z + x^3 - z^3 + 2z^2 - z, yz - y) \subseteq \mathbb{C}[x, y, z].$$

(a) Compute $\sqrt{J}$. *Hint:* Use Hilbert's Nullstellensatz.

(b) Determine the irreducible components of $V(J)$, and the minimal associated primes of J?

(c) Sketch the real variety $V(J) \cap \mathbb{R}^3$.

Exercise 2 (Varieties in linear algebra)**.** Let K be an infinite field. Which of the following subsets of $K^{n\times n}$ (identified with $\mathbb{A}^{n^2}$) are affine varieties?

(a) The set $\mathrm{Sym}(n)$ of symmetric matrices.

(b) The set $\mathrm{SL}(n)$ of matrices with determinant one.

(c) The set $X_{r,n}$ of matrices of rank at most r.

(d) The set $\mathrm{GL}(n)$ of invertible matrices.

Exercise 3 (Finite varieties)**.** Let $X \subseteq \mathbb{A}^n$ be an affine variety over a field K (not necessarily algebraically closed). Show the following statements:

(a) X is a singleton if and only if $A(X)$ is isomorphic to K.

(b) X is a finite set with cardinality r if and only if $A(X)$ is isomorphic to K^r. Conclude that finite varieties of the same cardinality are isomorphic.

(c) X is finite if and only if $A(X)$ is Artinian.

Exercise 4 (Properties of pullbacks)**.** Let $f\colon X \to Y$ be a morphism of affine varieties, and let $f^*\colon A(Y) \to A(X)$ be the pullback. Which of the following claims are true respectively false?

(a) f is surjective if f^* is injective

(b) f^* is injective if f is surjective

(c) f is injective if f^* is surjective

(d) f^* is surjective if f is injective

Exercise 5 (Finite morphisms)**.** Let K be an algebraically closed field.

(a) Let $f\colon X \to Y$ be a morphism of affine varieties. Show that for each $a \in X$, the preimage of the maximal ideal $I_X(\{a\})$ under the pullback $f^*\colon A(Y) \to A(X)$ is $I_Y(\{f(a)\})$.

(b) Show that a morphism of varieties $f\colon X \to Y$ is surjective with finite fibers if the pullback $f^*\colon A(Y) \to A(X)$ is finite and injective. *Hint:* Use "Lying Over" and [Gat13, Cor. 9.21(b)].

(c) Use the Noether normalization lemma (see [Gat13, Prop. 10.5]) to prove the following fact:

For each affine variety $X \subseteq \mathbb{A}^n$, there exists a nonnegative integer $d \leq n$ and a linear morphism $\Phi\colon X \to \mathbb{A}^d$ such that $\Phi^*\colon K[y_1, \ldots, y_d] \to A(X)$ is finite and injective.

(d) Find d and Φ with this property for the affine variety $X = V(x_1x_2) \subseteq \mathbb{A}^2_{\mathbb{C}}$.

(e) Use (b) and (c) to prove that every complex affine variety $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ with infinitely many points is unbounded with respect to the Euclidean metric. Is this true for real affine varieties?

[Gat13] Andreas Gathmann. *Commutative Algebra*, Class Notes TU Kaiserslautern, 2013. Available at https://agag-gathmann.math.rptu.de/de/commalg.php.

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