EXERCISES WEEK 2: THE ZARISKI TOPOLOGY AND DIMENSION

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Exercise 1 (The Zariski topology)**.**

(a) Determine the closure of the following sets:

$$\mathbb{Z} \text{ in } \mathbb{A}^1_{\mathbb{R}}, \quad \{(\cos(t), \sin(t), t)) : t \in \mathbb{R}\} \text{ in } \mathbb{A}^3_{\mathbb{R}}, \quad \mathrm{GL}(n) \text{ in } \mathbb{R}^{n\times n} \text{ (identified with } \mathbb{A}^{n^2}_{\mathbb{R}}) .$$

(b) Prove that for affine varieties $X, Y \subseteq \mathbb{A}^n$ it holds that $X \setminus Y$ is dense in X if and only if Y contains no irreducible components of X.

(c) Find a homeomorphism of affine varieties $f : X \to Y$ that is not an isomorphism.
Hint: Use Example 4.9 from the notes.

Exercise 2 (Irreducibility)**.**

(a) Show that if $f : X \to Y$ is a continuous map of topological spaces, and X is irreducible, then $f(X)$ is also irreducible.

(b) Let A be a subset of a topological space X. Show that A is irreducible if and only if the closure $\overline{A}$ in X is irreducible.

Exercise 3 (Images of morphisms)**.**

(a) Let $f : X \to Y$ be a morphism of affine varieties, and let $Z = \overline{\mathrm{im}(f)}$ be the closure of the image in Y. Prove that $I_Y(Z) = \ker(f^*)$ and that $A(Z) \cong \mathrm{im}(f^*)$.

(b) Compute the ideal of the closure of the image of the map $f : \mathbb{A}^1_K \to \mathbb{A}^3_K$ with $t \mapsto (t, t^2, t^3)$ over an infinite field K.

(c) A map $f : X \to Y$ of topological spaces is said to be *dominant* if $\mathrm{im}(f)$ is dense in Y.
Prove that a morphism $f : X \to Y$ of affine varieties is dominant if and only if f^* is injective.

Exercise 4 (Dimension)**.** Recall from the first exercise sheet the ideal

$$J = \langle x^3z^2 - 2x^3z + x^3 - z^3 + 2z^2 - z, yz - y\rangle \subseteq \mathbb{C}[x, y, z].$$

(a) What is the dimension of each of the irreducible components of the variety $X = V(J)$?

(b) What is the dimension of X?

(c) What is the local dimension of X at the points $(0, 0, 0)$ and $(1, 0, 1)$?

(d) Prove that if a morphism $f : X \to Y$ of affine varieties over an algebraically closed field is dominant and finite (in the sense that $f^* : A(Y) \to A(X)$ is finite) then $\dim(X) = \dim(Y)$.
Hint: Use the "lying over", "going up" and "incomparability" theorems for finite extensions.

(e) In the light of part (d), what is the d in a Noether normalization $X \to \mathbb{A}^d$?
Illustrate this by finding a Noether normalization of the variety X from part (a).

Exercise 5 (Hypersurfaces)**.** Let K be an algebraically closed field.

(a) Prove that $X = V(x_1x_4 - x_2x_3)$ is irreducible, with $\dim(X) = 3$.

(b) Prove that $Y = V_X(x_1, x_2)$ has codimension 1 in X.

(c) Prove that it is not possible to find any single $f \in A(X)$ such that $Y = V_X(f)$.

Comment: This exercise shows that affine varieties might contain hypersurfaces that are not defined by exactly one polynomial, contrary to the case when we consider hypersurfaces in $\mathbb{A}^n$.

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Exercise 6 (Varieties in linear algebra, revisited)**.** Let K be an algebraically closed field. For each of the following subsets of $K^{n\times n}$ from the exercise sheet in Week 1, determine whether it is **irreducible**, and what the **dimension** is.

(a) The set $\mathrm{Sym}(n)$ of symmetric matrices.

(b) The set $\mathrm{GL}(n)$ of invertible matrices.

(c) The set $\mathrm{SL}(n)$ of matrices with determinant one.

(d) The set $X_{r,n}$ of matrices of rank at most r.

Hint: Exercise 2 could be useful for proving irreducibility. It is also useful to remember that the dimension of an irreducible affine variety X is the same as for any nonempty open subset $U \subseteq X$.

Exercise 7 (Products of affine varieties)**.** Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.

(a) Show that $X \times Y$ is an affine variety in $\mathbb{A}^{n+m}$ under the identification $\mathbb{A}^n \times \mathbb{A}^m \leftrightarrow \mathbb{A}^{n+m}$.

(b) Show that the resulting (Zariski) topology on $X \times Y$ is not necessarily equal to the product topology inherited from X and Y.
Hint: Consider the diagonal $\Delta = \{(a,a) : a \in K\} \subseteq \mathbb{A}^1_K \times \mathbb{A}^1_K$ for an infinite field K.

(c) Prove that for any subsets $S \subseteq X$ and $T \subseteq Y$, it holds that $\overline{S \times T} = \overline{S} \times \overline{T}$.

(d) Show that $X \times Y$ is irreducible if X and Y are irreducible.

(e) Recall (or look up!) the universal property of a *product* in a category. Prove that $X \times Y$ with the canonical projections $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ satisfies the universal property of a product in the category of affine varieties.

(f) Show that $A(X \times Y)$ is isomorphic to $A(X) \otimes_K A(Y)$ as K-algebras.
Hint: Use that $(R_1/I_1) \otimes_K (R_2/I_2) \cong (R_1 \otimes_K R_2)/(I_1 \otimes_K R_2 + R_1 \otimes_K I_2)$.
Alternative hint: If you like category theory, you can instead try to show that both $A(X \times Y)$ and the tensor product satisfies the universal property of a *coproduct* in the category of reduced finitely generated K-algebras.

(g) Show that $\dim(X \times Y) = \dim(X) + \dim(Y)$ when K is algebraically closed.
Hint: Use Noether normalization.