EXERCISES WEEK 3: SHEAVES AND VARIETIES

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Exercise 1 (Localizations and regular functions)**.** Consider the affine variety

$$X = V\big((z-x^3)(z-1), (z-1)y\big) \subseteq \mathbb{A}^3_{\mathbb{C}}$$

from Exercise 4 of Week 2.

(a) Show that $\mathscr{O}_X(D(z-1)) \cong \mathbb{C}[x]_{x^3-1}$ and $\mathscr{O}_X(D(y)) \cong \mathbb{C}[x,y]_y$.

(b) Show the following characterizations of the stalks $\mathscr{O}_{X,(0,0,0)}$ and $\mathscr{O}_{X,(0,0,1)}$:
Every germ of $\mathscr{O}_{X,(0,0,0)}$ can be represented by $\big(D(f), \frac{g}{f}\big)$ for $f, g \in \mathbb{C}[x]$ with $f(0) \neq 0$.
Every germ of $\mathscr{O}_{X,(0,0,1)}$ can be represented by $\big(D(f), \frac{g}{f}\big)$ for $f, g \in \mathbb{C}[x,y]$ with $f(0,0) \neq 0$.

(c) Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $a \in X$. Prove that $\mathscr{O}_{X,a} \cong \mathscr{O}_{\mathbb{A}^n,a}/I(X)\mathscr{O}_{\mathbb{A}^n,a}$.
Find the stalks from (b) again using this result.

Hint: For (a) and (c), you can use that $R_T/IR_T \cong (R/I)_{\pi(T)}$ for any ring R, multiplicative subset $T \subseteq R$, and ideal I with quotient map $\pi \colon R \to R/I$.

Exercise 2 (Locally determined sections)**.**

(a) Let X be a topological space with a sheaf $\mathcal{F}$, let $U \subseteq X$ be an open subset, and let $\varphi, \psi \in \mathcal{F}(U)$ be sections. Show that if φ and ψ agree on all stalks (in the sense that $\overline{(U,\varphi)} = \overline{(U,\psi)}$ in $\mathcal{F}_a$ for all $a \in U$), then $\varphi = \psi$ in $\mathcal{F}(U)$.

(b) Prove that if X is an irreducible affine variety and $\mathcal{F} = \mathscr{O}_X$, then $\varphi = \psi$ in $\mathcal{F}(U)$ if φ and ψ agree on one stalk.

(c) Show that the property in (b) does not hold in general for an arbitrary sheaf on an arbitrary topological space.

Exercise 3. Which of the following ringed spaces are isomorphic?

(a) $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \subseteq \mathbb{A}^1_{\mathbb{C}}$

(b) $V(x\,y) \subseteq \mathbb{A}^2_{\mathbb{C}}$

(c) $V(x^2+y^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$

(d) $V(y^2-x^3-x^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$

(e) $V(y-x^2, z-x^3) \setminus \{0\} \subseteq \mathbb{A}^3_{\mathbb{C}}$

(f) $V(x^2-y^2-1) \subseteq \mathbb{A}^2_{\mathbb{C}}$

Exercise 4 (The algebraic torus)**.** Let K be an algebraically closed field K, and let $K^* = K \setminus \{0\}$.

(a) Show that every regular function in $\mathscr{O}_{\mathbb{A}^n}((K^*)^n)$ is given by a Laurent polynomial.

(b) Give an example of an embedded affine variety that is isomorphic to $(K^*)^n$ as a ringed space.

(c) Let X be an affine variety, and let $A(X)^*$ be the set of invertible elements in $A(X)$. Give a bijection between the set of ringed space morphisms $X \to (K^*)^n$ and $(A(X)^*)^n$.

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Exercise 5. Let K be an algebraically closed field.

(a) Let $X \subseteq \mathbb{A}^n$ be an irreducible affine variety, and let $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$. Prove that for every irreducible component Z of $X \cap V(f_1, \ldots, f_r)$, it holds that

$$\dim(Z) \geq \dim(X) - r.$$

(b) Let $X, Y \subseteq \mathbb{A}^n$ be irreducible affine varieties. Prove that for every irreducible component Z of $X \cap Y$, it holds that

$$\dim(Z) \geq \dim(X) + \dim(Y) - n\,.$$

Hint: Show that $X \cap Y \cong (X \times Y) \cap \Delta$ for the diagonal $\Delta = \{(x, x) : x \in \mathbb{A}^n\} \subseteq \mathbb{A}^n \times \mathbb{A}^n$.

(c) Show that (a) and (b) are not true in general without the irreducibility hypotheses.

(d) Optional challenge: Find examples where the inequalities in (a) and (b) are strict for some irreducible components but not all of them.

Exercise 6. Let X and Y be varieties over an algebraically closed field K. Recall that if both X and Y are affine, we have seen that we have a bijection

$$\{\text{Morphisms } X \to Y\} \longrightarrow \{K\text{-algebra homomorphisms } \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)\}, \quad f \mapsto f^*.$$

Determine whether this is still a bijection under the following weaker assumptions:

(a) X is an arbitrary variety and Y is affine;

(b) Y is an arbitrary variety and X is affine.