EXERCISES WEEK 4: PROJECTIVE VARIETIES

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If not stated otherwise, we will assume that we are working over an algebraically closed field K.

Exercise 1 (Lines in projective space)**.** Recall that a (projective) *line* in $\mathbb{P}^n$ is a projective variety of the form $\mathbb{P}(V)$ for some 2-dimensional linear subspace $V \subseteq K^{n+1}$.

(a) Show that any two lines L_1 and L_2 in $\mathbb{P}^2$ intersect.

(b) Find the intersection between $L_1 = V_p(x_1 - x_2)$ and $L_2 = V_p(x_0 + x_1 - x_2)$ in $\mathbb{P}^2$ over $\mathbb{C}$.

(c) Let L_1 and L_2 be two disjoint lines in $\mathbb{P}^3$ and let $p \in \mathbb{P}^3 \setminus (L_1 \cup L_2)$. Show that there exists a unique line $L \subseteq \mathbb{P}^3$ that intersects p, L_1 and L_2.

(d) Find such an L from (c) for $L_1 = V_p(x_1, x_2)$, $L_2 = V_p(x_0 - x_1, x_3)$, and $p = (1 : 0 : 1 : 0)$ in $\mathbb{P}^3$ over $\mathbb{C}$.

(e) Draw a sketch of the real part of the objects in (b) and (d) in U_0 (by setting $x_0 = 1$).

Hint: For (a) and (c), it might be useful to remember the *Grassmann formula* from linear algebra, which says that $\dim(V \cap W) = \dim(V) + \dim(W) - \dim(V + W)$ for linear subspaces $V, W \subseteq K^n$.

Exercise 2 (Equations for linear projective varieties)**.**

(a) Find a generating set for the homogeneous ideal $I(L) \subseteq K[x_0, \ldots, x_n]$ of the linear projective variety $L = \mathbb{P}(V) \subseteq \mathbb{P}^n$ in terms of a basis $\{v_1, \ldots, v_{r+1}\}$ of the linear subspace $V \subseteq K^{n+1}$.

(b) As an example of (a), find a generating set of $I(L)$ for the projective line $L = \mathbb{P}(V) \subseteq \mathbb{P}^3_{\mathbb{C}}$ with $V = \mathrm{span}_{\mathbb{C}}\{(1, 1, 1, 1), (1, 2, 3, 4)\} \subseteq \mathbb{C}^4$.

Exercise 3 (Products of varieties)**.**

(a) Let X and Y be two varieties. Show that $X \times Y$ is also a variety.

(b) Show that $\dim(X \times Y) = \dim(X) + \dim(Y)$.

(c) Show that if X and Y are irreducible, then $X \times Y$ is also irreducible.

Hint: Recall Exercise 7 of Week 2. For (b) you can use that $\dim U = \dim X$ for any open subset U of an irreducible variety X (see the summary notes of Week 2 for a proof).

Exercise 4 (Dimension and intersection theory)**.** In this problem, we will use the convention that $\dim(\varnothing) = -1$ (i.e., the dimension of the empty projective variety is -1).

(a) Let $X \subseteq \mathbb{P}^n$ be a projective variety. Show that $\dim(C(X)) = \dim(X) + 1$. Deduce that if $Y \subseteq \mathbb{A}^{n+1}$ is an affine variety that is a cone, then $\dim(\mathbb{P}(Y)) = \dim(Y) - 1$.

(b) Let $X = \mathbb{P}(V)$ and $Y = \mathbb{P}(W)$ be linear projective varieties in $\mathbb{P}^n$ defined by linear subspaces $V, W \subseteq K^{n+1}$. Show that $\dim(X \cap Y) = \dim(X) + \dim(Y) - \dim(\mathbb{P}(V + W))$.

(c) Prove the following important generalization of Exercise 1(a):

If $X, Y \subsetneq \mathbb{P}^n$ are projective varieties such that $\dim(X) + \dim(Y) \geq n$, then $X \cap Y \neq \varnothing$.

(d) Use (c) to show that $\mathbb{P}^m \times \mathbb{P}^n$ is not isomorphic to $\mathbb{P}^{m+n}$ if both n and m are nonzero.

Exercise 5 (Projective closure).

(a) Compute the projective closure and the points at infinity of the following affine varieties:

(i) $V_a(x_2^2 - x_1^3) \subseteq \mathbb{A}^2$

(ii) $V_a(x_2 - x_1^2, x_2 + x_1 - 2) \subseteq \mathbb{A}^2$

(iii) $V_a(x_2(x_3 - 1), (x_3 - x_1^3)(x_3 - 1)) \subseteq \mathbb{A}^3$.

(b) Let $X \subseteq \mathbb{A}^n$ be an affine variety with projective closure $\overline{X} \subseteq \mathbb{P}^n$. Show $I_p(\overline{X}) = (I_a(X))^h$.

Hint: Use the projective Nullstellensatz. A useful fact that you might want to prove as a lemma is that $\sqrt{J^h} = (\sqrt{J})^h$ for ideals $J \subseteq K[x_1, \ldots, x_n]$.

Exercise 6 (Saturation). Let $I, J \subset K[x_1, \ldots, x_n]$ be two ideals. Recall that the *saturation* of I *with respect to* J is the ideal:

$$I : J^\infty := \{f \in K[x_1, \ldots, x_n] \ : \ \text{there exists } m > 0 \text{ such that } fJ^m \subseteq I\}.$$

(a) Show that the associated primes of $I : J^\infty$ are precisely the associated primes of I not containing $\sqrt{J}$. *Hint:* Consider a primary decomposition of I.

(b) Show that $V_a(I : J^\infty) = \overline{V_a(I) \setminus V_a(J)}$, and that for affine varieties $X, Y \subseteq \mathbb{A}^n$, it holds that $I_a(X \setminus Y) = I_a(X) : I_a(Y)^\infty$.

Let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal. We will refer to the saturation of I with respect to the irrelevant ideal $B = \langle x_0, \ldots, x_n \rangle \subset K[x_0, \ldots, x_n]$ simply as *the saturation*, and denote it

$$I^{\text{sat}} := I : B^\infty.$$

We say that I is *saturated* if $I^{\text{sat}} = I$.

(c) Show that if I is radical, then I^{sat} is homogeneous and radical.

(d) Show that every radical ideal is saturated. Find an example of a saturated ideal that is not radical.

(e) Show that $V_p(I^{\text{sat}}) = V_p(I)$.

(f) Let $f_1, \ldots, f_r$ be homogeneous polynomials in $K[x_0, \ldots, x_n]$ of degrees $d_1, \ldots, d_r$, respectively. Fix an integer $d \geq \max\{d_1, \ldots, d_r\}$, and let I be the ideal generated by all monomial multiples of the generators f_i of degree d, i.e.,

$$I := \langle \{ g f_i : g \in \text{Mon}(d - d_i, n + 1) \}_{i=1,\ldots,r} \rangle,$$

where $\text{Mon}(\ell, n + 1)$ is the set of all monomials of degree ℓ in $x_0, \ldots, x_n$.

Show that $\langle f_1, \ldots, f_r \rangle^{\text{sat}} = I^{\text{sat}}$.

Note that this shows that for any projective variety X, there exists a homogeneous ideal J with $X = V_p(J)$ that can be generated by polynomials that all have the same degree.