EXERCISES WEEK 5: MORE PROJECTIVE GEOMETRY AND GRASSMANNIANS

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If not stated otherwise, we will assume that we are working over an algebraically closed field K.

Exercise 1. Let $S = \bigoplus_{d\geq 0} S_d$ and $T = \bigoplus_{d\geq 0} T_d$ be graded rings. A *homomorphism of graded rings* $\varphi\colon S \to T$ is a ring homomorphism such that for some fixed $k \in \mathbb{Z}_{\geq 0}$, it holds that

$$\varphi(S_d) \subseteq T_{kd} \quad \text{for all } d \geq 0.$$

We say that φ is *degree-preserving* if $k = 1$.

(a) Show that if $\varphi\colon S \to T$ is an isomorphism of graded rings, then φ must be degree-preserving.

(b) Show that if two projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ have isomorphic homogeneous coordinate rings, then there exists an isomorphism $f\colon X \to Y$.

(c) Consider the degree-2 Veronese embedding

$$\nu_{1,2}\colon \mathbb{P}^1 \to \mathbb{P}^2, \quad (x_0 : x_1) \mapsto (x_0^2 : x_0x_1 : x_1^2),$$

whose image is the conic $C = V_p(y_0y_2 - y_1^2)$. Show that the pullback $\nu_{1,2}^*\colon S(C) \to \mathbb{C}[x_0, x_1]$ is not an isomorphism of graded rings.

(d) Is it true that if two projective varieties are isomorphic, then their coordinate rings are also isomorphic?

Exercise 2 (The Veronese embedding)**.**

(a) Give finitely many quadratic polynomials whose zero locus is the image of the degree-d Veronese embedding $\nu_{n,d}\colon \mathbb{P}^n \to \mathbb{P}^N$ where $N = \binom{n+d}{n} - 1$.

(b) Show that any projective variety $X \subseteq \mathbb{P}^n$ can be embedded into $\mathbb{P}^N$ for some $N \geq 0$ such that it becomes the zero locus of quadratic and linear polynomials.

Exercise 3 (Enumerative geometry)**.** Recall that $G(2, 4)$ can be regarded as the set of lines in $\mathbb{P}^3$, and let $\iota\colon G(2, 4) \to \mathbb{P}^5$ be the Plücker embedding.

(a) Show that $\iota(G(2, 4)) = V_p(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})$.

(b) Fix a line $\ell \subseteq \mathbb{P}^3$, and consider the set $\mathcal{X}_\ell$ of lines in $\mathbb{P}^3$ that intersect ℓ, viewed as a subset of $G(2, 4)$. Show that $\iota(\mathcal{X}_\ell) = \iota(G(2, 4)) \cap V_p(f)$ for a linear homogeneous polynomial f. Illustrate your answer by finding such an f when $\ell = \mathbb{P}(\text{span}\{(1, 0, 0, 0), (0, 1, 0, 0)\})$.
Hint: Recall (or look up) Laplace expansion by complementary minors.

(c) Show that there exist infinitely many lines in $\mathbb{P}^3$ passing through three lines $\ell_1, \ell_2, \ell_3 \subseteq \mathbb{P}^3$.

(d) Challenge: How many lines do you expect pass through four general lines $\ell_1, \ell_2, \ell_3, \ell_4 \subseteq \mathbb{P}^3$?

Exercise 4 (Incidence correspondences)**.** Show that the following are projective varieties:

(a) $\mathcal{Z}_{k,n} = \{(L, a) \in G(k, n+1) \times \mathbb{P}^n : a \in L\}$ (this is an example of an *incidence correspondence*, i.e., a set of tuples of geometric objects subject to some membership condition).

(b) The union $\mathcal{L}(X)$ of all lines in $\mathbb{P}^n$ that intersect a projective variety $X \subseteq \mathbb{P}^n$.
Hint: View $\mathcal{L}(X)$ as a projection of a certain incidence correspondence.

(c) The **join** $\mathcal{J}(X, Y)$ of two disjoint varieties $X, Y \subseteq \mathbb{P}^n$, defined as the union of all lines intersecting both X and Y.

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Exercise 5 (Properties of projective spaces)**.** Let m and n be positive integers.

(a) Show that any morphism $f\colon \mathbb{P}^n \to \mathbb{P}^m$ is of the form $x \mapsto (f_0(x) : \cdots : f_m(x))$ for homogeneous polynomials $f_i \in K[x_0, \ldots, x_n]$ of the same degree satisfying $V_p(f_0, \ldots, f_m) = \varnothing$.

Hint: It may help to understand the proofs of Proposition 7.2 and Lemma 7.4 in the notes.

(b) Show that if $n > m$ and $Y_0, \ldots, Y_m$ are hypersurfaces in $\mathbb{P}^n$, then $Y_0 \cap \cdots \cap Y_m \neq \varnothing$.

Hint: Consider Exercise 5 from Week 3, and Exercise 4(c) from Week 4.

(c) Show that if $n > m$, then every morphism $f\colon \mathbb{P}^n \to \mathbb{P}^m$ is constant.

Remark: Parts (a) and (b) are independent of each other. Part (c) is a consequence of the statements in (a) and (b).