EXERCISES WEEK 6: SMOOTHNESS AND BLOW-UPS

CARLES CHECA, ELISENDA FELIU, OSKAR HENRIKSSON

Throughout K denotes an algebraically closed field of any characteristic.

Exercise 1 (Projective Jacobi criterion)**.**

(a) (Euler identity) Show that if $f \in K[x_0, \dots, x_n]$ is a homogeneous polynomial of degree d, then

$$d\, f = \sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} .$$

(b) Let $X \subseteq \mathbb{P}^n$ be a projective variety with homogeneous ideal $I(X) = \langle f_1, \dots, f_r \rangle \subseteq K[x_0, \dots, x_n]$. Show that X is smooth at $a \in X$ if and only if the rank of the Jacobian matrix

$$J(a) := \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{i,j} \in K^{r \times (n+1)}$$

is at least $n - \operatorname{codim}_X \{a\}$.

(c) Let $C = V_p(x_2^2 x_3 - x_1^2(x_3 - x_1)) \subseteq \mathbb{P}^2$ over $K = \mathbb{C}$. Show that C has exactly one singular point (and find it).

(d) For the curve C in (c), show that the affine curve in the affine chart where $x_1 \neq 0$ and the affine curve in the affine chart where $x_2 \neq 0$ are both smooth.

(This illustrates that a smooth affine variety can become singular after considering its projective closure. In that case, the singular points are all at infinity).

Exercise 2 (Smooth hypersurfaces in $\mathbb{P}^n$)**.** Let $n \geq 2$. Show:

(a) Every smooth hypersurface in $\mathbb{P}^n$ is irreducible.

(b) A general hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ is smooth (hence irreducible).

Specifically, using the coefficients of the polynomials, identify the vector space of all homogeneous polynomials of degree d in $x_0, \dots, x_n$ modulo scalars with the projective space $\mathbb{P}^N_{\mathbb{C}}$ where $N = \binom{n+d}{n}$. Let $U_d \subseteq \mathbb{P}^N_{\mathbb{C}}$ consist of the coefficients of the polynomials f such that f is irreducible and $V_p(f)$ is smooth. Show that U_d is dense and open.

(c) Find U_d for $d = 2$ and $n = 1$.

*Note: The complement of the set U_d in (b) is an irreducible projective variety Δ_d of codimension 1 in $\mathbb{P}^N_{\mathbb{C}}$. Points in Δ_d are the coefficients for which the polynomial is singular. The polynomial defining Δ_d is called the **discriminant** of degree d homogeneous polynomials.*

Exercise 3. For integers $0 \leq r < n$, consider the affine variety

$$X_{r,n} = \{A \in K^{n \times n} : \operatorname{rank}(A) \leq r\} .$$

Show that the set of singular points of $X_{r,n}$ is $X_{r-1,n}$ if $r > 0$.

Hint: Start with $r = n - 1$. Consider coordinates x_{ij}, $i, j = 1, \dots, n$, corresponding to the entries of the matrices $A \in K^{n \times n}$. Use (without proof) that the polynomials obtained from the $(r+1) \times (r+1)$ minors of an $n \times n$ matrix generate $I(X_{r,n})$. Recall that we have already seen in Week 2 that $X_{r,n}$ is irreducible of dimension $r^2 - 2r(n - r)$.

Exercise 4. Let $C = V_p(x_2^2 x_3 - x_1^2(x_3 - x_1)) \subseteq \mathbb{P}^2$ over $K = \mathbb{C}$. Show that blowing up C at the only singular point you found in Exercise 1 produces a smooth variety.

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Exercise 5. For every integer $\ell \geq 0$, consider the irreducible affine variety $X_\ell = V(x_2^2 - x_1^{2\ell+1}) \subseteq \mathbb{A}^2$.

(a) Show that if $\ell > 0$, then $(0,0)$ is the only singular point of X_ℓ.

(b) Show that $X_{\ell'}$ and X_ℓ are not isomorphic if $\ell' \neq \ell$.
Hint: Consider the blow-up of X_ℓ at $(0,0)$.

(c) Are $X_{\ell'}$ and X_ℓ for $\ell' \neq \ell$ birationally equivalent?

Exercise 6. Let $\widetilde{\mathbb{A}}^3$ be the blow-up of $\mathbb{A}^3$ at the x_3-axis $V(x_1, x_2)$ and let $\pi\colon \widetilde{\mathbb{A}}^3 \to \mathbb{A}^3$ be the blow-up map.

(a) Show that $\widetilde{\mathbb{A}}^3 = \{((x_1, x_2, x_3), (y_1 : y_2)) \in \mathbb{A}^3 \times \mathbb{P}^1 : x_2 y_1 = x_1 y_2\}$.

(b) Compute the exceptional set $\pi^{-1}(V(x_1, x_2))$.

(c) Consider the strict transform of a line L intersecting $V(x_1, x_2)$ and different from the x_3-axis. When does a pair of such strict transforms intersect in the exceptional set of $\widetilde{\mathbb{A}}^3$? Discuss the geometric meaning of your answer.

(d) Show that the variety obtained after blowing up the Whitney umbrella $X = V(x_2^2 - x_3 x_1^2)$ at the x_3-axis is smooth.

Exercise 7. Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subsetneq X$ be irreducible, closed subsets, no one contained in the other. Moreover, let $\widetilde{X}$ be the blow-up of X at the ideal $I(Y_1) + I(Y_2)$. Show that the strict transforms of Y_1 and Y_2 are disjoint.