EXERCISES WEEK 7: HILBERT FUNCTIONS AND BÉZOUT'S THEOREM

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Throughout we work over an algebraically closed field K, unless otherwise stated.

Exercise 1 (Hilbert functions)**.** Let $R = K[x_0, \dots, x_n]$.

(a) Let $\ell \in R_1$ be a linear form. Determine the Hilbert function of $\langle \ell \rangle$.

(b) What is the Hilbert function of $\langle \ell^k \rangle$ for $k \in \mathbb{Z}_{>0}$?

(c) What is the Hilbert function of $\langle \ell_1^k, \ell_2^k \rangle$ for linearly independent forms $\ell_1, \ell_2 \in R_1$?

Hint: You can first show that $K[x_0, \dots, x_n]/\langle x_0^k \rangle \cong K[x_0, \dots, x_n]/\langle \ell^k \rangle$.

Exercise 2. Compute the Hilbert polynomial and degree of the following projective varieties:

(a) $V_p(x_1 x_2) \subseteq \mathbb{P}^2$.

(b) $V_p(x_0, x_1) \cup V_p(x_2, x_3) \subseteq \mathbb{P}^3$.

(c) $V_p(x_1, x_2) \cup V_p(x_1, x_3) \subseteq \mathbb{P}^3$.

(d) The image of $\mathbb{P}^1 \to \mathbb{P}^3$, $(s : t) \mapsto (s^3 : s^2 t : s t^2 : t^3)$ (the *twisted cubic*).

Exercise 3 (Veronese embedding)**.** The goal of this exercise is to prove that the degree of a projective variety $X \subseteq \mathbb{P}^n$ is not an isomorphism invariant, but rather depends on the embedding of X in a projective space.

(a) Show that the degree of $\mathbb{P}^n$ is 1.

(b) Show that the degree of the Veronese embedding $\nu_{n,d}(\mathbb{P}^n) \subseteq \mathbb{P}^{N-1}$, where $N = \binom{n+d}{n}$, is d^n.

Exercise 4 (Multiplicity)**.** Compute the local multiplicity of the following ideals in $K[x, y]$ at the given points:

(a) $I = \langle x^4, y^4 \rangle$ at $(0, 0)$.

(b) $I = \langle x, y \rangle^4$ at $(0, 0)$.

(c) $I = \langle x, y \rangle^4$ at $(0, 1)$.

(d) $I = \langle y^2, xy - y, x^3 - x^2 \rangle$ at $(0, 0)$.

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Exercise 5 (Bézout in the torus)**.** Fix integers $d_1, \ldots, d_n \in \mathbb{Z}_{\geq 0}$. The goal of this exercise will be to prove that a system of polynomials

$$g_1(x_1, \ldots, x_n) = \cdots = g_n(x_1, \ldots, x_n) = 0$$

with degrees $\deg(g_i) = d_i$ has precisely $d_1 \cdots d_n$ solutions, all contained in $(\mathbb{C}^*)^n$, if the coefficients are chosen generically. We will show this by homogenizing the problem, and showing that there exists an open subset $U \subseteq \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_n}$ for $N_i = \binom{n+d_i}{n} - 1$ such that

$$(\mathbf{c}_1, \ldots, \mathbf{c}_n) \in U \implies \#V_p(f_1, \ldots, f_n) = d_1 \cdots d_n \text{ and } V_p(f_1, \ldots, f_n) \cap V_p(x_0 \cdots x_n) = \varnothing\,,$$

where $f_i = \sum_{\alpha \in \mathcal{M}_{n,d_i}} c_{i,\alpha} x^\alpha$, $\mathbf{c}_i = (\ldots c_{i,\alpha} \ldots)_{\alpha \in \mathcal{M}_{n,d_i}}$, and $\mathcal{M}_{n,d_i} = \{\alpha \in \mathbb{Z}_{\geq 0}^{n+1} : \sum_{i=0}^n \alpha_i = d_i\}$.

(a) Show that there is an open set in $\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_n}$ of coefficients of polynomials $f_1, \ldots, f_n$ such that $V_p(f_1, \ldots, f_n)$ is finite. *Hint:* Show that we generically have a complete intersection.

(b) Show that there is an open set $W \subseteq \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_{n-1}}$ such that the ideal $\langle f_{1|x_i=0}, \ldots, f_{n-1|x_i=0} \rangle$ corresponds to a complete intersection in $\mathbb{P}^n \cap \{x_i = 0\} \cong \mathbb{P}^{n-1}$.

(c) Show that if $I \subseteq K[x_0, \ldots, x_n]$ is a homogeneous ideal with $V_p(I) \subseteq \mathbb{P}^n$ finite and non-empty. Show that there is an open subset $V \subseteq \mathbb{P}^{\binom{n+d}{n}-1}$ such that if f is a polynomial of degree d with coefficients in V, then $V_p(I) \cap V_p(f) = \emptyset$.

(d) Derive the result.

Exercise 6 (Pascal's theorem)**.** Let f be an irreducible homogeneous polynomial of degree two. Pick six distinct points $P_1, \ldots, P_6$ on the irreducible projective conic $V_p(f)$ in $\mathbb{P}^2$, and let $\overline{P_i P_j}$ be the line through P_i and P_j for $i, j \in \{1, \ldots, 6\}$. Pascal's theorem states that the points

$$P = \overline{P_1P_2} \cap \overline{P_4P_5}\,, \qquad Q = \overline{P_2P_3} \cap \overline{P_5P_6}\,, \qquad R = \overline{P_3P_4} \cap \overline{P_1P_6}$$

lie on a line. To prove the theorem, consider the following steps:

(a) Show that the two hypersurfaces

$$X_1 = \overline{P_1P_2} \cup \overline{P_3P_4} \cup \overline{P_5P_6} \qquad X_2 = \overline{P_2P_3} \cup \overline{P_4P_5} \cup \overline{P_1P_6}$$

intersect in the points $P_1, \ldots, P_6, P, Q, R$.

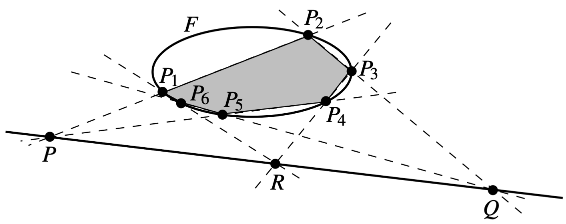
(b) Let $g_1, g_2 \in K[x_0, x_1, x_2]_3$ be such that $X_1 = V_p(g_1)$ and $X_2 = V_p(g_2)$. Given a point $S \in V_p(f) \setminus \{P_1, \ldots, P_6\}$ consider a cubic curve of the form

$$g = \lambda_1 g_1 + \lambda_2 g_2$$

for some $(\lambda_1, \lambda_2) \neq (0, 0)$ by imposing that $S \in V_p(g)$. Show that $V_p(g) \cap V_p(f)$ contains at least seven points. Does this contradict Bézout's theorem?

(c) Use that f is irreducible to deduce that $V_p(g)$ contains a unique line ℓ.

(d) Show that P, Q, R are contained in ℓ.



Exercise 7 (Bézout for isolated solutions)**.** Let $X \subseteq \mathbb{P}^n$ be a non-empty pure-dimensional projective variety of dimension d. The goal of this exercise is to show that if L is a linear variety of codimension d, then the number of 0-dimensional components of $X \cap L$ is at most $\deg(X)$.

(a) Show that if $I \subseteq J$ are homogeneous ideals and $\dim V_p(I) = \dim V_p(J)$, then $\deg(J) \leq \deg(I)$.

(b) Let $X \subseteq \mathbb{P}^n$ be a projective variety of pure dimension d, and let $\ell \in K[x_0, \ldots, x_n]$ be a homogeneous linear form. Let Y_1 be the union of all irreducible components Z of X such that $\ell \notin I(Z)$, and let $X_1 = Y_1 \cap V_p(\ell)$. Prove that X_1 is the union of all irreducible components of $X \cap V_p(\ell)$ that have dimension $d - 1$, and that $\deg(X_1) = \deg(Y_1) \leq \deg(X)$.

(c) Prove the result in the introduction of the exercise.