Lecture 9: Grassmannians

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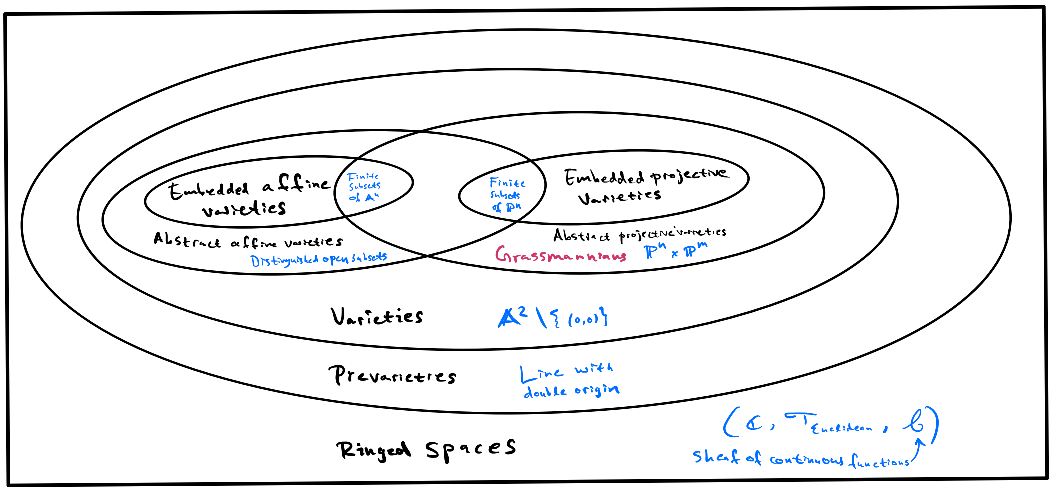
Wednesday, Week 5

2025-05-21

§ Quick recap

Spaces



Properties

- Irreducibility
- Dimension
- Smoothness of varieties (week 6)
- Degree of projective varieties (week 7)

§ Introduction

Last week: $\mathbb{P}^{n-1} = \dfrac{K^n \setminus \{0\}}{\text{Scaling}} \longleftrightarrow \left\{ \begin{matrix} \text{1D linear} \\ \text{subspaces of } K^n \end{matrix} \right\}$ with affine open charts

$$U_i = \{x \in \mathbb{P}^{n-1} : x_i \neq 0\} = \{(x_1 : \cdots : 1 : \cdots x_n) : x_1, \ldots, x_n \in K\} \cong \mathbb{A}^{n-1}$$

Goal for today:

Realize the Grassmannian $G(r,n) := \left\{ \begin{matrix} r\text{-dimensional linear} \\ \text{subspaces of } K^n \end{matrix} \right\}$ as a projective variety.

We will do it in two steps:

1. Give $G(r,n)$ the structure of a prevariety by gluing affine open charts.
2. Show that $G(r,n)$ is isomorphic to a closed subset of projective space.

Remark: Gathmann directly identifies $G(r,n)$ with a closed subset of projective space, and uses the language of exterior algebras to make the presentation more streamlined.

We will take a slightly more hands-on approach here.

For the exercises and exam, you are free to choose the perspective you like the most!

Motivation: We have a bijection $G(r,n) \longleftrightarrow \left\{ \begin{matrix} (r-1)\text{-dimensional linear} \\ \text{projective varieties in } \mathbb{P}^{n-1} \end{matrix} \right\}$

$$L \longmapsto \mathbb{P}(L)$$

Hence, $G(r,n)$ is the natural habitat for enumerative geometry.

Examples of enumerative problems in $G(2,4)$:

- How many lines intersect four general lines in $\mathbb{P}^3$?

  Answered by Schubert (1879). The formalization of Schubert's techniques was part of Hilbert's 15th problem.

  Exercise 3 of this week!

- How many lines are contained in a smooth cubic surface in $\mathbb{P}^3$?

  Answer by Cayley-Salmon (1849): 27. See Chapter 11 of Gathmann's notes. This is not part of the course, but a nice application of the theory.

⚠ Because of this projective interpretation, some authors (and Macaulay2) write $G(r-1,n-1)$ for what we call $G(r,n)$.

§ Construction.

Observation: $G(r,n) = \left\{ \text{rowspan}(M) : \begin{matrix} M \in K^{r\times n} \\ \text{rank}(M) = r \end{matrix} \right\}$

$$= \bigcup_{I \in \binom{[n]}{r}} \left\{ \text{rowspan}(M) : \begin{matrix} M \in K^{r\times n} \\ \det(M_I) \neq 0 \end{matrix} \right\} = \bigcup_{I \in \binom{[n]}{r}} \underbrace{\left\{ \text{rowspan}(M) : \begin{matrix} M \in K^{r\times n} \\ M_I = E_r \end{matrix} \right\}}_{\mathcal{U}_I}$$

Notation: $[n] = \{1,2,\dots,n\}$ M_I = submatrix of M given by the columns with indices in I for $M \in K^{r\times n}$ and $I \in \binom{[n]}{r}$

$\binom{[n]}{r} = \{ I \subseteq [n] : \#I = r \}$ E_r = $r\times r$ identity matrix

We have a bijection

$$\Phi_I : \underset{\cong \mathbb{A}^{r(n-r)}}{K^{r\times(n-r)}} \longrightarrow \mathcal{U}_I \qquad \text{where}$$

$$C \longmapsto \operatorname{rowspan}(M_I(C)) \qquad M_I(C) := \left(\begin{array}{c}\text{the matrix } M \in K^{r\times n} \\ \text{with } M_I = E_r \text{ and} \\ M_{[n]\setminus I} = C\end{array}\right)$$

This allows us to view $\mathcal{U}_I$ as an abstract affine variety, which is irreducible and of dimension $r(n-r)$.

We will now turn $G(r,n)$ into a prevariety by gluing the sets $\mathcal{U}_I$ (cf. Construction 5.6 in the notes), via the identity maps as transition maps:

$$\mathcal{U}_I \cap \mathcal{U}_J \xrightarrow[\text{identity}]{f_{I,J}} \mathcal{U}_J \cap \mathcal{U}_I .$$

Since we use identity maps for the gluing, we only need to check two things:

① $\mathcal{U}_I \cap \mathcal{U}_J$ is open in $\mathcal{U}_I$:

$$\Phi_I^{-1}(\mathcal{U}_I \cap \mathcal{U}_J) = \{ C \in K^{r\times(n-r)} : \det(M_I(C))_J \neq 0 \}$$

is clearly open in $K^{r\times(n-r)}$, so $\mathcal{U}_I \cap \mathcal{U}_J$ is open in $\mathcal{U}_I$.

② $f_{I,J}$ is a morphism:

We have a commutative diagram

$$
\begin{tikzcd}
\mathcal{U}_I \cap \mathcal{U}_J \arrow[r, "f_{I,J}"] & \mathcal{U}_J \cap \mathcal{U}_I \\
\Phi_I^{-1}(\mathcal{U}_I \cap \mathcal{U}_J) \arrow[u, "\Phi_I"] \arrow[r] & \Phi_J^{-1}(\mathcal{U}_J \cap \mathcal{U}_I) \arrow[u, "\Phi_J"'] \\
c \arrow[r, mapsto] & (M_I(c))_J^{-1}\,(M_I(c))_{[n]\setminus J}
\end{tikzcd}
$$

Note that $(M_I(c))_J^{-1}$ is a well-defined regular function on this open subset of $k^{r\times(n-r)}$ since $\det(M_I(c))_J \neq 0$

well-defined morphism on $\Phi_I^{-1}(\mathcal{U}_I \cap \mathcal{U}_J)$

Example: $\mathcal{U}_{\{1,2\}}$ and $\mathcal{U}_{\{1,3\}}$ in $G(2,4)$

$$
\begin{tikzcd}
\operatorname{rowspan}\begin{pmatrix} 1 & 0 & c_{11} & c_{12} \\ 0 & 1 & c_{21} & c_{22} \end{pmatrix} \arrow[rrr, mapsto] & & & \operatorname{rowspan}\left(\begin{pmatrix} 1 & c_{11} \\ 0 & c_{21} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & c_{11} & c_{12} \\ 0 & 1 & c_{21} & c_{22} \end{pmatrix} \right) \arrow[ddd, mapsto] \\
& \mathcal{U}_{\{1,2\}} \cap \mathcal{U}_{\{1,3\}} \arrow[r, "\text{identity}"] & \mathcal{U}_{\{1,3\}} \cap \mathcal{U}_{\{1,2\}} & \\
& \Phi_{\{1,2\}}^{-1}(\mathcal{U}_{\{1,2\}} \cap \mathcal{U}_{\{1,3\}}) \arrow[u, "\Phi_{\{1,2\}}"] \arrow[r] & \Phi_{\{1,3\}}^{-1}(\mathcal{U}_{\{1,3\}} \cap \mathcal{U}_{\{1,2\}}) \arrow[u, "\Phi_{\{1,3\}}"'] & \\
\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \arrow[uuu, mapsto] \arrow[rrr, mapsto] & & & \begin{pmatrix} 1 & c_{11} \\ 0 & c_{21} \end{pmatrix}^{-1} \begin{pmatrix} 0 & c_{12} \\ 1 & c_{22} \end{pmatrix}
\end{tikzcd}
$$

Conclusion: The above construction turns $G(r,n)$ into a prevariety with affine open covering $G(r,n) = \bigcup_{I \in \binom{[n]}{r}} \mathcal{U}_I$.

Remark: For $r=1$, we recover $\mathbb{P}^{n-1}$ as $G(1,n)$.

§ Basic properties

Prop. $G(r,n)$ is irreducible.

Proof. By construction, $G(r,n) = \bigcup_{I \in \binom{[n]}{r}} \mathcal{U}_I$ is a union of irreducible open sets that pairwise overlap, which gives that $G(r,n)$ is irreducible. $\square$

In both these proofs, we use facts from the document "Addendum on irreducibility and dimension" on Absalon.

Prop. $\dim\big(G(r,n)\big) = r(n-r)$

Pf. By irreducibility, we get

$$\dim\big(G(r,n)\big) = \dim(\mathcal{U}_I) = \dim\left(k^{r\times(n-r)}\right) = r(n-r). \quad \square$$

Prop. We have an isomorphism

$$f : G(r,n) \longrightarrow G(n-r,n)$$
$$L \longmapsto L^\perp$$

$L^\perp := \{x \in k^n : \sum_{i=1}^n x_i y_i = 0 \;\; \forall y \in L\}$.

Pf. $\boxed{\text{Well-defined}}$ It's a standard linear algebra exercise to show that $L^\perp \in G(n-r,n)$.

$\boxed{\text{Bijective}}$ The inverse is

$$g : G(n-r,n) \longrightarrow G(r,n)$$
$$L \longmapsto L^\perp.$$

$\boxed{\text{Morphism}}$ To see that f (and analogously g) are morphisms, we check what it does on the charts $\mathcal{U}_I$.

For notational simplicity, take $I = [r] \in \binom{[n]}{r}$, and let

$$\operatorname{rowspan}(E_r \mid c) \in \mathcal{U}_I \subseteq G(r,n)$$

We know from linear algebra that

$$\operatorname{rowspan}(E_r \mid c)^\perp = \{ x \in k^n : (E_r \mid c)x = 0 \}$$
$$= \ker(E_k \mid c) = \operatorname{columnspan}\begin{pmatrix} -c \\ E_{n-r} \end{pmatrix}$$
$$= \operatorname{rowspan}(-c^T \mid E_{n-r}) \in \mathcal{U}_{[n]\setminus I} \subseteq G(n-r,n),$$

So on the level of the charts, we have

$$\begin{tikzcd}
\operatorname{rowspan}(E_r \mid c) \arrow[rrr, mapsto] & & & \operatorname{rowspan}(-c^T \mid E_{n-r}) \arrow[ddd, mapsto] \\
& \mathcal{U}_I \arrow[r, "f"] & \mathcal{U}_{[n]\setminus I} \arrow[d, "\Phi^{-1}_{[n]\setminus I}"] & \\
& k^{r\times(n-r)} \arrow[u, "\Phi_I"] \arrow[r] & k^{(n-r)\times r} & \\
c \arrow[uuu, mapsto] \arrow[rrr, mapsto] & & & -c^T
\end{tikzcd}$$

Which clearly is a morphism $k^{r\times(n-r)} \longrightarrow k^{(n-r)\times r}$. $\square$

§ Projectiveness

Left to do: show that $G(r,n)$ is an (abstract) projective variety

Key construction: The Plücker embedding is defined by

$$\begin{aligned} Pl: \quad G(r,n) &\longrightarrow \mathbb{P}^{\binom{n}{r}-1} \\ \operatorname{rowspan}(M) &\longmapsto \left(\det(M_I)\right)_{I\in\binom{[n]}{r}} \end{aligned}$$

← With homogeneous coordinates x_I indexed by $I \in \binom{[n]}{r}$ in some arbitrary order (e.g. lexicographic).

Example: Let $L = \operatorname{rowspan}\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in G(2,4)$. Then

$$\begin{array}{cccccccccccccc} & \{1,2\} & & \{1,3\} & & \{1,4\} & & \{2,3\} & & \{2,4\} & & \{3,4\} & \\ Pl(L) = \Bigg(& \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & : & \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} & : & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & : & \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & : & \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & : & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & \Bigg) \end{array}$$

$$= (1 : 0 : 1 : -2 : -1 : 2) \in \mathbb{P}^5$$

Plan: (1) Show that Pl is a well-defined map.

(2) Show that Pl is a morphism.

(3) Show that $Pl(G(r,n)) \subseteq \mathbb{P}^{\binom{n}{r}-1}$ is closed.

(4) Show that Pl gives an isomorphism $G(r,n) \to Pl(G(r,n))$.

This is similar to how we used the Segre embedding to show that $\mathbb{P}^n \times \mathbb{P}^m$ is projective!

Well-definedness:

- Since $\operatorname{rank}(M) = r$, at least one r-minor of M is nonzero, so $\left(\det(M_I)\right)_{I\in\binom{[n]}{r}}$ is a point in $\mathbb{P}^{\binom{n}{r}-1}$
- Suppose $\operatorname{rowspan}(M) = \operatorname{rowspan}(N)$ for some $M, N \in K^{r\times n}$ of rank r. Then $\exists T \in GL(r)$ such that $M = TN$, and it holds that
$$\det(M_I) = \det\left((TN)_I\right) = \det(T N_I) = \det(T)\det(N_I) \quad \text{for each } I \in \binom{[n]}{r},$$

which gives

$$Pl(\operatorname{rowspan}(M)) = \left(\det(M_I)\right)_{I\in\binom{[n]}{r}} = \left(\overbrace{\det(T)}^{\neq 0}\det(N_I)\right)_{I\in\binom{[n]}{r}} = \left(\det(N_I)\right)_{I\in\binom{[n]}{r}} = Pl(\operatorname{rowspan}(N)). \qquad \square$$

Morphism: We check this on the charts $\mathcal{U}_I$ of $G(r,n)$:

For notational simplicity, suppose $I = [r] \in \binom{[n]}{r}$.

Then we obtain a commutative diagram

$$
\begin{tikzcd}
\mathrm{rowspan}(E_k|C) \arrow[rrr, mapsto] & & & \left(\det(E_k|C)_J\right)_{J\in\binom{[n]}{k}} \\
& \mathcal{U}_I \arrow[r, "Pl"] & \mathbb{P}^{\binom{n}{k}-1} & \\
& K^{r\times(n-r)} \arrow[u, "\Phi_I"] \arrow[r] & W_I := \{x\in\mathbb{P}^{\binom{n}{r}-1} : x_I \neq 0\}. \arrow[u, hook] & \\
C \arrow[uuu, mapsto] \arrow[rr, mapsto] & & \left(\det(E_k|C)_J\right)_{J\in\binom{[n]}{k}} \arrow[uuur, bend right] &
\end{tikzcd}
$$

Clearly a morphism on affine varieties

Closedness of the Image: We will prove that

$$Pl(G(r,n)) = \underbrace{V_P\left(g_{I,J} : I \in \binom{[n]}{r-1},\ J \in \binom{[n]}{r+1} \right)}_{X \subseteq \mathbb{P}^{\binom{n}{r}-1}}$$

Where
$$g_{I,J} = \sum_{j \in J \setminus I} (-1)^{s_{I,J}(j)} x_{I\cup\{j\}}\, x_{J\setminus\{j\}}$$

for $s_{I,J}(j) = \#\{i \in I : j < i\} + \#\{l \in J : l < j\}$

Terminology: Polynomials vanishing on $Pl(G(r,n)) \subseteq \mathbb{P}^{\binom{n}{r}-1}$ are called **Plücker relations** of $G(r,n)$.

$\boxed{Pl(G(r,n)) \subseteq X}$ Consider $M = \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \\ v_{r1} & \cdots & v_{rn} \end{pmatrix} \in K^{r\times n}$ with $\operatorname{rank}(M)=r$.

Then, for each $I \in \binom{[n]}{r-1}$ and $J \in \binom{[n]}{r+1}$, we obtain

$$g_{I,J}(Pl(\operatorname{rowspan}(M))) = \sum_{j\in J\setminus I} (-1)^{s_{I,J}(j)} \det(M_{I\cup\{j\}}) \det(M_{J\setminus\{j\}})$$

Reorder columns of $M_{I\cup\{j\}}$ so that M_j appears last and note that $\det(M_{I\cup\{j\}})=0$ for $j\in I\cap J$, so that the sum can be extended to all of J

Suppose $J=\{j_1<\cdots<j_{r+1}\}$

r×r matrix

$$= \sum_{j\in J} (-1)^{s_{I,J}(j)} (-1)^{\#\{i\in I : j<i\}} \det(M_I | M_j) \det(M_{J\setminus\{j\}}) = \sum_{k=1}^{r+1} (-1)^k \det(M_I|M_{j_k}) \det(M_{J\setminus\{j_k\}})$$

jth column of M

(r−1) × (r−1) matrix

Laplace expansion along last column
$$= \sum_{k=1}^{r+1}(-1)^k\left(\sum_{l=1}^{r} (-1)^{r+l} \det(M_{\underbrace{[r]\setminus\{l\}}_{\text{rows}},\underbrace{I}_{\text{columns}}}) v_{l,j_k}\right) \det(M_{J\setminus\{j_k\}})$$

Rearrange terms
$$= \sum_{l=1}^{r} (-1)^{r+l} \det(M_{[r]\setminus\{l\},I}) \left(\sum_{k=1}^{r+1} (-1)^k v_{l,j_k} \det(M_{J\setminus\{j_k\}})\right)$$

(r+1) × (r+1) matrix

Reversed Laplace expansion
$$= \sum_{l=1}^{r} (-1)^{r+l} \det(M_{[r]\setminus\{l\},I}) \underbrace{\det\begin{pmatrix} (v_l)_J^T \\ M_J \end{pmatrix}}_{=0} = 0.$$

because it has a repeated row

$\boxed{X \subseteq Pl(G(r,n))}$ Let $p \in X \cap \{x_I \neq 0\} \subseteq \mathbb{P}^{\binom{n}{r}-1}$ for some fixed $I \in \binom{[n]}{r}$.

We now want to construct a matrix $M_p^I \in K^{r\times n}$ with rank r such that $Pl(\operatorname{rowspan}(M_p^I)) = p$

After rescaling of p, we can assume $p_I = 1$, and set

Kronecker delta

$$(M_p^I)_{kl} = \begin{cases} \delta_{kl} & \text{if } l\in I \\ (-1)^{\#\{i\in I \text{ between } i_k \text{ and } l\}} p_{I\cup\{l\}\setminus\{i_k\}} & \text{if } l \in [n]\setminus I \end{cases}$$

One can now check that

$$Pl(\operatorname{rowspan}(M_p^I)) = \left(\det((M_p^I)_J)\right)_{J\in\binom{[n]}{r}} = (p_J)_{J\in\binom{[n]}{r}} = p. \qquad \square$$

Example: For $G(2,4)$, there is a total of $4 \cdot 4 = 16$ choices of $I \in \binom{[4]}{1}$, $J \in \binom{[4]}{3}$.

Taking $I = \{1\}$ and $J = \{2,3,4\}$ gives the Plücker relation

$$g_{I,J} = x_{12}x_{34} + (-1)x_{13}x_{24} + x_{14}x_{23} \qquad (*)$$

We will see in Exercise 3 that this single quadric is enough to cut out $Pl(G(2,4)) \subseteq \mathbb{P}^5$.

Next, suppose $p = (1 : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) \in X \cap \{x_I \neq 0\}$ for $I = \{1,2\}$.

The construction in the proof gives the matrix

$$M_p^I = \begin{bmatrix} 1 & 0 & -p_{23} & -p_{24} \\ 0 & 1 & p_{13} & p_{14} \end{bmatrix}$$

Let's check the minors:

$$Pl\left(\operatorname{rowspan}(M_p^I)\right) = \left(\underset{\{1,2\}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} : \underset{\{1,3\}}{\begin{vmatrix} 1 & -p_{23} \\ 0 & p_{13} \end{vmatrix}} : \underset{\{1,4\}}{\begin{vmatrix} 1 & -p_{24} \\ 0 & p_{14} \end{vmatrix}} : \underset{\{2,3\}}{\begin{vmatrix} 0 & -p_{23} \\ 1 & p_{13} \end{vmatrix}} : \underset{\{2,4\}}{\begin{vmatrix} 0 & -p_{24} \\ 1 & p_{14} \end{vmatrix}} : \underset{\{3,4\}}{\begin{vmatrix} -p_{23} & -p_{24} \\ p_{13} & p_{14} \end{vmatrix}} \right)$$

$$= \left(1 : p_{13} : p_{14} : p_{23} : p_{14} : -p_{14}p_{23} + p_{13}p_{24}\right)$$

$$-p_{14}p_{23} + p_{13}p_{24} \underset{\text{By the Plücker relation } (*)}{=} p_{12}p_{34} \underset{p_{12}=1}{=} p_{34}$$

Example: For $G(3,6)$, our construction gives rise to $\binom{6}{2} \cdot \binom{6}{4} = 225$ Plücker relations. However, a Macaulay2 calculation reveals that we can get away with 35 polynomials to generate the ideal of $Pl(G(3,6)) \subseteq \mathbb{P}^{20}$. Note that this is still more than

$$\operatorname{codim}\left(Pl(G(3,6))\right) = 20 - \dim(G(3,6)) = 20 - 3 \cdot (6-3) = 11.$$

For $p = (1 : p_{124} : \cdots : p_{456}) \in X \cap \{x_{123} \neq 0\}$, we have $M_p^{123} = \begin{bmatrix} 1 & 0 & 0 & p_{234} & p_{235} & p_{236} \\ 0 & 1 & 0 & -p_{134} & -p_{135} & -p_{136} \\ 0 & 0 & 1 & p_{124} & p_{125} & p_{126} \end{bmatrix}$.

Isomorphism:

The above construction gives a morphism $X \cap \{x_I \neq 0\} \to \mathcal{U}_I$, $p \longmapsto \mathrm{rowspan}(M_p^I)$

of affine varieties, which is the inverse of $Pl|_{\mathcal{U}_I} : \mathcal{U}_I \longrightarrow X \cap \{x_I \neq 0\}$.

These local inverses glue to a global inverse:

If $p \in X \cap \{x_I \neq 0\} \cap \{x_J \neq 0\}$ for $I, J \in \binom{[n]}{r}$, then

$$(M_p^I)_J^{-1} M_p^I = M_p^J \quad \text{where } (M_p^I)_J \in GL(r)$$

which gives $\mathrm{rowspan}(M_p^I) = \mathrm{rowspan}(M_p^J)$. $\square$

Conclusion: The Plücker embedding gives an isomorphism

$$G(r,n) \xrightarrow[\cong]{Pl} Pl(G(r,n)) = V_p\left(g_{I,J} : \begin{array}{l} I \in \binom{[n]}{r-1} \\ J \in \binom{[n]}{r+1} \end{array} \right) \subseteq \mathbb{P}^{\binom{n}{r}-1}.$$

In particular, $G(r,n)$ is an (abstract) projective variety.

This has many nice theoretical consequences!

For instance, $G(r,n)$ is complete, and therefore behaves well with respect to projections. We will exploit this in Exercise 4 to form interesting varieties such as the join of two disjoint varieties $X, Y \subseteq \mathbb{P}^n$

$J(X,Y) :=$ Union of all lines that intersect both X and Y.

The Plücker relations themselves are also interesting.

In Exercise 3 we will see an example of how problems in enumerative geometry can be translated to problems of counting solutions to polynomial systems involving Plücker relations of an appropriate Grassmannian!